Exam I Review Math 261

Name: Solutions Section: ____________

Parametric Equations

1. A particle follows the trajectory

$$\begin{cases} x(t) = 7t + 9 \\ y(t) = -t^2 + 12t \end{cases}$$

with t in seconds and distance in centimeters.

(a) What is the particles maximum height?

max height ONLY when $\frac{dy}{dt} = -2t + 12 = 0$

ONLY when $t = 6$

max height IS $y(6) = -(6)^2 + 12(6)$

$= -36 + 72$

$= 36$

(b) When does the particle hit the ground, and how far from the origin does it land?

hits ground when $y(t) = 0 = -t^2 + 12t$

$0 = t(-t+12)$

when $t = 0$ or $t = 12$

start end

hits ground AT $x(12) = 7(12) + 9 = 72$

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2. Some modern cell phones are able to track your change in x and y position without using a GPS, by means of their built-in accelerometers and a little bit of Calculus 1.

   Suppose you are walking directly up a hill, and that your position function is given by $c(t) = (7t - 10, -t^3 + 30t^2)$.

   (a) Find the slope of the hill at $t = 5$ during your walk, without eliminating the parameter.

$$\text{slope of hill} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3t^2 + 60t}{7}$$

$$\text{when } t=5, \text{ slope} = \frac{-3\cdot(5)^2 + 60\cdot 5}{7} = \frac{-75 + 300}{7} = \frac{225}{7}$$

   (b) Eliminate the parameter to find the height of the hill as a function of your horizontal position.

$$x(t) = 7t - 10$$

$$x + 10 = 7t$$

$$\frac{x+10}{7} = t$$

$$y(t) = -t^3 + 30\cdot t^2$$

$$y = -\left(\frac{x+10}{7}\right)^3 + 30\cdot\left(\frac{x+10}{7}\right)^2$$

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3. Let $c(t) = (t^2 - 9, t^2 - 8t)$

(a) Find the equation of the tangent line at $t = 4$.

$$y = m(x - x_1) + y_1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-8}{2t-9}$$

when $t = 4 \Rightarrow m = \frac{2 \cdot 4 - 8}{2 \cdot 4 - 9} = \frac{0}{-1} = 0$

$x_1 \doteq x(4) = 4^2 - 9 = 16 - 9 = 7$

$y_1 = y(4) = 4^2 - 8 \cdot 4 = -16$

$$y = 0 \cdot (x - 7) + (-16)$$

$$y = -16$$

(b) Find the points where the tangent has slope $\frac{1}{2}$.

find when $\frac{dy}{dx} = \frac{1}{2} = \frac{2t-8}{2t-9}$

$$2t - 9 = 4t - 16$$

$$7 = 2t$$

$$t = \frac{7}{2}$$

this is at $x\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 9 = \frac{49}{4} - 9$

$y\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 8 \cdot \frac{7}{2} = \frac{49}{4} - 28$

(c) Find the points where the tangent is horizontal or vertical.

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 2t - 8$
$= 2(t-4)$

tangent is horizontal when $\frac{dx}{dt} \neq 0$, $\frac{dy}{dt} = 0$

$2t \neq 0$ $\quad 2(t-4) = 0$

when $t = 4$

at $x(4) = 4^2 - 9$
$y(4) = 4^2 - 8 \cdot 4$

at $(7, -16)$

tangent is vertical when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$

$2t = 0$ $\quad 2(t-4) \neq 0$

when $t = 0$

at $x(0) = 0^2 - 9 = -9$
$y(0) = 0^2 - 8 \cdot 0 = 0$

at $(-9, 0)$

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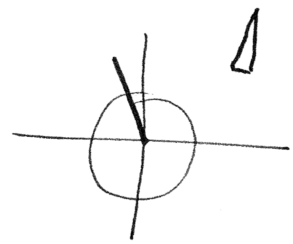
4. Find an equation for the line tangent to the curve $c(t) = \left(4\sin\left(\frac{t}{2}\right), 4\cos\left(\frac{t}{2}\right)\right)$ at the point when $t = 4\pi/3$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \cdot \cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}}{4 \cdot \left(-\sin\left(\frac{t}{2}\right)\right) \cdot \frac{1}{2}} = \frac{2 \cdot \cos\left(\frac{t}{2}\right)}{-2 \cdot \sin\left(\frac{t}{2}\right)}$$

when $t = \frac{4\pi}{3}$, $m = \frac{\cancel{2} \cdot \cos\left(\frac{4\pi/3}{2}\right)}{-\cancel{2} \cdot \sin\left(\frac{4\pi/3}{2}\right)} = \frac{\cos\left(\frac{2\pi}{3}\right)}{-\sin\left(\frac{2\pi}{3}\right)}$



$$= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$x_1 = 4 \cdot \sin\left(\frac{4\pi/3}{2}\right) = 4 \cdot \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$y_1 = 4 \cdot \cos\left(\frac{4\pi/3}{2}\right) = 4 \cdot \cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$

$$y = \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right) + \frac{-1}{2}$$

5. Find an equation for the line tangent to the curve $c(t) = (3e^{-t}, 3e^{2t})$ at the point when $t = 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{3} \cdot e^{-t} \cdot (-1)}{\cancel{3} \cdot e^{2t} \cdot 2}$$

when $t = 0$, $m = \frac{e^{-0}(-1)}{e^{2 \cdot 0} \cdot 2} = \frac{1 \cdot (-1)}{1 \cdot 2} = \frac{-1}{2}$

$x_1 = x(0) = 3 \cdot e^{-0} = 3$

$y_1 = y(0) = 3 \cdot e^{2 0} = 3$

$$y = \frac{-1}{2}(x - 3) + 3$$

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6. Find the points where the curve $c(t) = (t^2 - 9, t^3 - 12t)$ has horizontal tangents and vertical tangents.

See Homework 1, question D

7. Obtain a Cartesian equation for the path by eliminating the parameter

$$\begin{cases} x(\theta) = 1 + 3\sin(\theta) \\ y(\theta) = 2 - 4\cos(\theta) \end{cases}$$

See Homework 1, Question H

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8. Obtain a Cartesian equation for the path by eliminating the parameter

$$\begin{cases} x(t) = \cos^2(2t) \\ y(t) = \cos(t) \end{cases}$$

See Homework 1, Question I

9. Obtain a Cartesian equation for the path by eliminating the parameter

$$\begin{cases} x(t) = 1 + \sin^2(t) \\ y(t) = \cos(t) \end{cases}$$

See Homework 1, Question J

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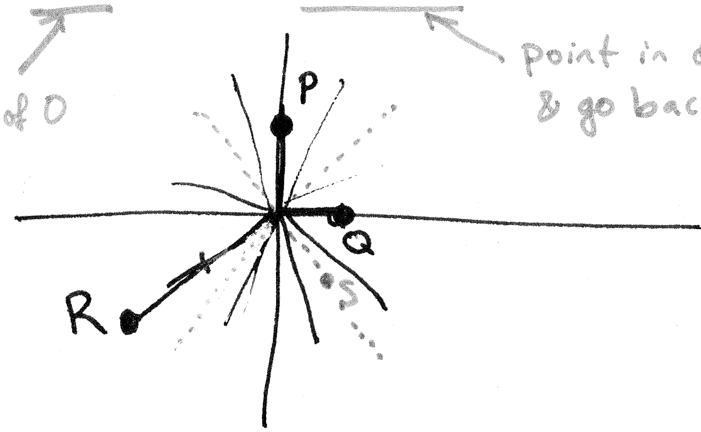
Polar Coordinates, Equations, and Area

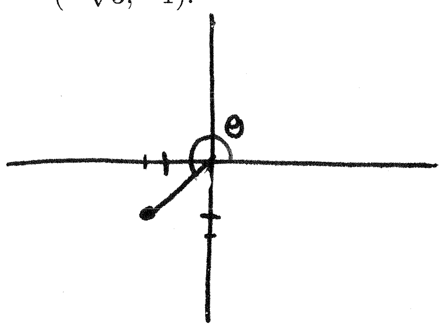
1. Plot and label the points with the following polar coordinates:
$P = (1, \frac{\pi}{2})$, $Q = (\frac{\pi}{2}, 0)$, $R = (2, \frac{7\pi}{6})$, $S = (-1, \frac{3\pi}{4})$.

$\pi/2$ in angle of 0

point in direction $\frac{3\pi}{4}$ & go backward 1.



2. Find a polar coordinate representation of the point with Cartesian (rectangular) coordinates $(-\sqrt{3}, -1)$.



$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4}$

$r = 2$

$\tan(\theta) = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$

know $\tan(\pi/6) = \frac{1}{\sqrt{3}}$

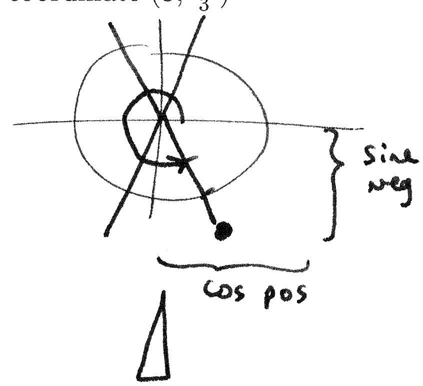
choose $\theta = \frac{7\pi}{6}$

Polar coordinate $\left(2, \frac{7\pi}{6}\right)$

3. Find the Cartesian (rectangular) coordinates for the point with polar coordinate $(3, \frac{5\pi}{3})$

$x = r \cdot \cos\theta = 3 \cdot \cos\left(\frac{5\pi}{3}\right) = 3 \cdot \frac{1}{2}$

$y = r \cdot \sin\theta = 3 \cdot \sin\left(\frac{5\pi}{3}\right) = 3 \cdot \frac{\sqrt{3}}{2}$



Cartesian coordinate

$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

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4. Obtain a polar function which graphs the Cartesian equation

$$xy^2 = 1$$

$$(r \cdot \cos\theta)(r \cdot \sin\theta)^2 = 1$$

$$r \cdot \cos\theta \cdot r^2 \cdot \sin^2\theta = 1$$

$$r^3 = \frac{1}{\cos\theta \cdot \sin^2\theta}$$

$$r = \sqrt[3]{\frac{1}{\cos\theta \cdot \sin^2\theta}}$$

<u>know</u>

$$r^2 = x^2 + y^2$$

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

5. Find a Cartesian equation for the graph of the polar function

$$r = \sin^2(\theta)$$

$$r^2 \cdot r = r^2 \cdot \sin^2\theta$$

$$r^3 = (r \cdot \sin\theta)^2$$

$$\left(\sqrt{x^2+y^2}\right)^3 = y^2$$

<u>know</u>

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2+y^2}$$

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

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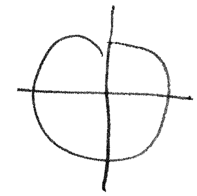
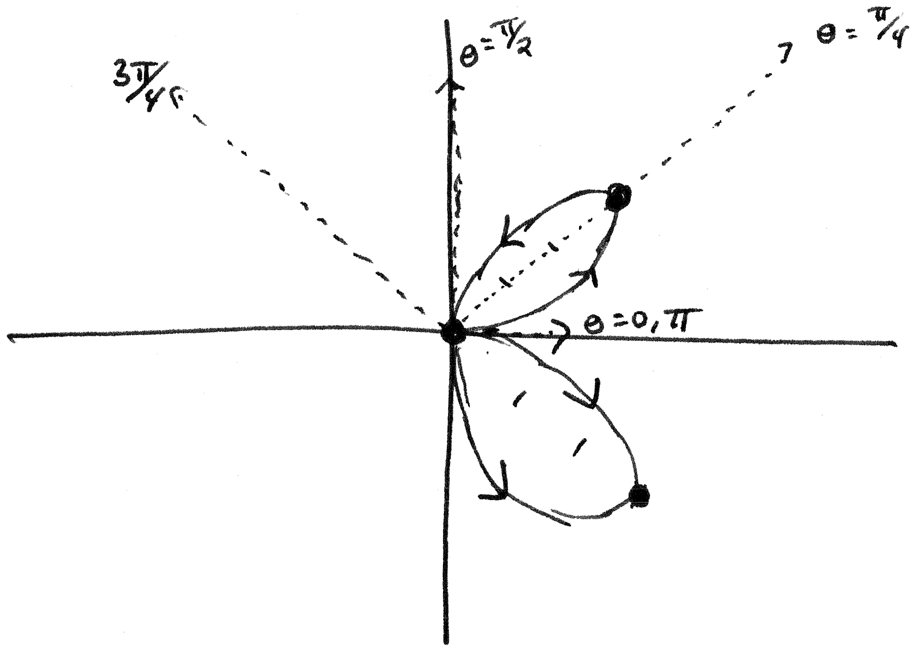


6. Fill in the following table of values for the function $r = 3\sin(2\theta)$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	$3\cdot\sin(0)$ $= 0$	$3\cdot\sin(2\cdot\frac{\pi}{4})$ $=3\cdot\sin(\pi/2)$ $= 3$	$3\cdot\sin(2\cdot\frac{\pi}{2})$ $=3\cdot\sin(\pi)$ $=0$	$3\cdot\sin(2\cdot\frac{3\pi}{4})$ $=3\cdot\sin(\frac{3\pi}{2})$ $=-3$	$3\cdot\sin(2\pi)$ $=3\cdot 0$ $=0$

Use it to sketch the ***restriction*** of the polar curve to θ in $[0, \pi]$.



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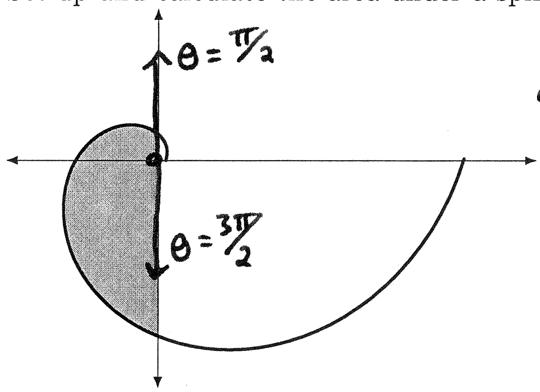
7. Set up and calculate the area under a spiral $r = \theta^2 + 1$ sketched below.



$$\text{area} = \int_{\pi/2}^{3\pi/2} \frac{1}{2}\left(\theta^2+1\right)^2 d\theta$$

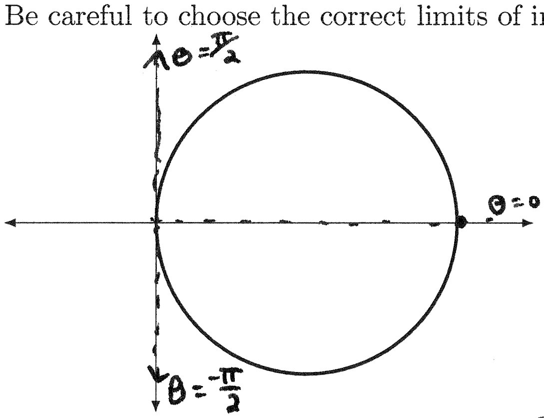
$$= \int_{\pi/2}^{3\pi/2} \frac{1}{2}\left(\theta^4+2\theta^2+1\right) d\theta$$

$$= \left[\frac{1}{2}\left(\frac{\theta^5}{5} + \frac{2\theta^3}{3} + \theta\right)\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{2}\left(\frac{\left(\frac{3\pi}{2}\right)^5}{5} + \frac{2\left(\frac{3\pi}{2}\right)^3}{3} + \left(\frac{3\pi}{2}\right)\right) - \frac{1}{2}\left(\frac{\left(\frac{\pi}{2}\right)^5}{5} + \frac{2\left(\frac{\pi}{2}\right)^3}{3} + \frac{\pi}{2}\right)$$

DONE!

8. Calculate the area of the circle $r = 4\cos(\theta)$ as an integral in polar coordinates. Be careful to choose the correct limits of integration.



$$\text{area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(4\cdot\cos\theta\right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{2}}\cdot \cancel{16}\cdot\cos^2\theta \, d\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= \int_{-\pi/2}^{\pi/2} \overset{4}{\cancel{8}}\cdot\frac{1+\cos(2\theta)}{\cancel{2}} d\theta = \left[4\left(\theta + \frac{\cos(2\theta)}{2}\right)\right]_{-\pi/2}^{\pi/2}$$

$$= 4\left(\frac{\pi}{2} + \frac{\cos\left(2\frac{\pi}{2}\right)}{2}\right) - 4\left(\frac{-\pi}{2} + \frac{\cos\left(2\frac{-\pi}{2}\right)}{2}\right) = 2\pi + \cancel{(-2)} - (-2\pi) - \cancel{2(-1)}$$

$$= 4\pi$$

θ $\cos(\pi) = -1$ $\cos(-\pi) = -1$

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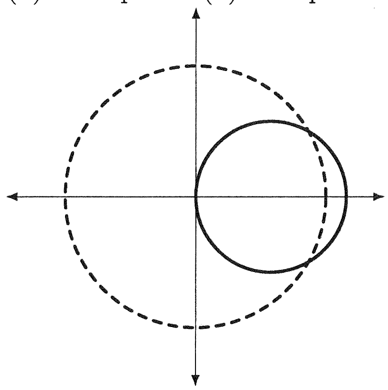
9. Sketch the area of the region inside the curve $r = 2\cos(\theta)$ and outside the circle $r = \sqrt{3}$. (1) Set up and (2) Compute the integral, being careful to select your limits of integration.



See Homework 2 Question 14A

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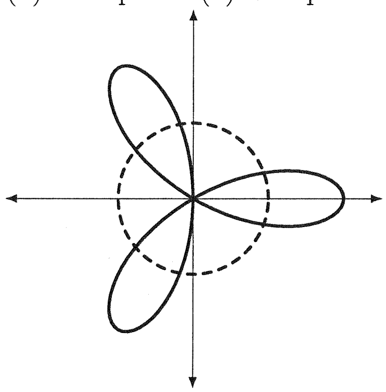
10. Sketch the area of the region inside one leaf of the rose $r = \cos(3\theta)$ and outside the circle $r = \frac{1}{2}$.

    (1) Set up and (2) Compute the integral, being careful to select your limits of integration.



See Homework 2 Question 14B

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Vectors in 2D and 3D

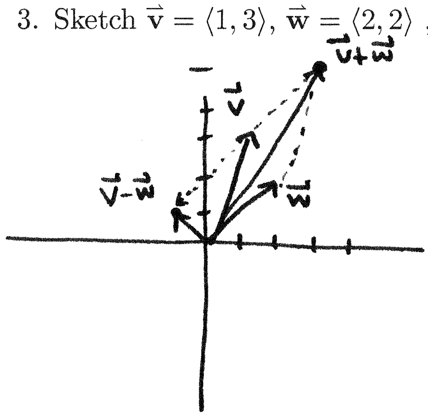
1. Find the components of $\overrightarrow{PQ}$ where P = (2, 3), Q = (1, 4).

$$\overrightarrow{PQ} = \langle 1-2, 4-3 \rangle$$
$$= \langle -1, 1 \rangle$$

2. Find the components of $\overrightarrow{PQ}$ where P = (2, 1, -2), Q = (-1, 2, 1).

$$\overrightarrow{PQ} = \langle -1-2, 2-1, 1-(-2) \rangle$$
$$= \langle -3, 1, 3 \rangle$$

3. Sketch $\vec{v} = \langle 1, 3 \rangle$, $\vec{w} = \langle 2, 2 \rangle$, $\vec{v} + \vec{w}$, $\vec{v} - \vec{w}$



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4. Find the length of the vector $\vec{\mathbf{u}} = \langle 1, 1 \rangle$

$$\|\vec{u}\| = \sqrt{1^2+1^2} = \sqrt{2}$$

5. Find the length of the vector $\vec{\mathbf{v}} = \langle 1, 2, -1 \rangle$

$$\|\vec{v}\| = \sqrt{1^2+2^2+(-1)^2}$$

$$= \sqrt{1+4+1}$$

$$= \sqrt{6}$$

6. Find the magnitude of the vector from P = (1,-1,1) to Q = (3,2,2).

$$\vec{PQ} = \langle 3-1, 2-(-1), 2-1 \rangle$$

$$= \langle 2, 3, 1 \rangle$$

$$\|\vec{PQ}\| = \sqrt{2^2+3^2+1^2}$$

$$= \sqrt{4+9+1}$$

$$= \sqrt{14}$$

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7. Find the unit vector $\vec{\mathbf{e}}_{\mathbf{v}}$ in the direction of $\vec{\mathbf{v}} = \langle 1, \sqrt{3} \rangle$

$$\|\vec{v}\| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\vec{e_v} = \frac{1}{\|\vec{v}\|} \cdot \vec{v} = 2 \cdot \langle 1, \sqrt{3} \rangle$$

$$\vec{e_v} = \langle 2, 2\sqrt{3} \rangle$$

8. Find the unit vector $\vec{\mathbf{e}}_{\mathbf{v}}$ in the direction of $\vec{\mathbf{v}} = \langle 1, -2, 3 \rangle$

$$\|\vec{v}\| = \sqrt{1^2 + (-2)^2 + 3^2}$$

$$= \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\vec{e_v} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

$$= \frac{1}{\sqrt{14}} \langle 1, -2, 3 \rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

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9. Find a parametrization for the line which passes through P = (1, 0, 2), with direction vector $\vec{v} = \langle 1, 0, -2 \rangle$

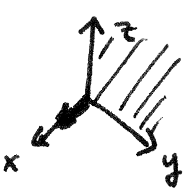
$\vec{p} = \overrightarrow{OP} = \langle 1, 0, 2 \rangle$

$F(t) = \vec{p} + t\vec{v}$

$F(t) = \langle 1, 0, 2 \rangle + t \cdot \langle 1, 0, -2 \rangle$

10. Find a parametrization for the line which passes through P = (1, 1, 0), which is perpendicular to the $y - z$ plane.

$\vec{p} = OP = \langle 1, 1, 0 \rangle$



$\langle 1, 0, 0 \rangle$ is $\perp$ to y-z plane

$F(t) = \langle 1, 1, 0 \rangle + t \cdot \langle 1, 0, 0 \rangle$

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The Dot Product

know $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$

1. Compute the dot product between the vectors $\langle 2, 1, -1\rangle$ and $\langle -1, 1, -1\rangle$. Are the vectors orthogonal? If not, is the angle between them acute or obtuse?

$$\langle 2,1,-1\rangle\cdot\langle -1,1,-1\rangle = 2(-1) + 1\cdot 1 + (-1)(-1) = -2 + 1 + 1 = 0$$

$\cos\theta = 0$

$\Rightarrow$ ~~angle~~ $\boxed{\text{the vectors are orthogonal } (\perp)}$

2. Compute the dot product between the vectors $\langle 2, -1, 3\rangle$ and $\langle 3, 1, -2\rangle$. Are the vectors orthogonal? If not, is the angle between them acute or obtuse?

$$\langle 2,-1,3\rangle\cdot\langle 3,1,-2\rangle = 2\cdot 3 + (-1)\cdot 1 + 3\cdot(-2) = 6 - 1 - 6 = -1$$

$\cos\theta < 0$

cos is neg

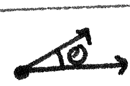
$\Rightarrow$

$\boxed{\text{the angle is obtuse}}$

3. Compute the dot product between the vectors $\langle 3, 1, 2\rangle$ and $\langle 2, 0, -1\rangle$. Are the vectors orthogonal? If not, is the angle between them acute or obtuse?

$$\langle 3,1,2\rangle\cdot\langle 2,0,-1\rangle = 3\cdot 2 + 1\cdot 0 + 2\cdot(-1) = 6 + 0 - 2 = 4$$

$\cos\theta > 0$



cos is pos

$\boxed{\text{the angle is acute}}$

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4. Find the component of $\vec{\mathbf{u}} = \langle 2, 1, -1 \rangle$ in the direction of the vector $\vec{\mathbf{v}} =< 1, \sqrt{3} >$, as well as the projection of $\vec{\mathbf{u}}$ in the direction of $\vec{\mathbf{v}}$.

5. Find the component of $\vec{\mathbf{u}} = \langle 2, -1, 3 \rangle$ in the direction of the vector $\vec{\mathbf{v}} =< 3, 0, 4 >$, as well as the projection of $\vec{\mathbf{u}}$ in the direction of $\vec{\mathbf{v}}$.